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# Stochastic dilations of the Bloch equations in boson and fermion noise

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**Abstract.** Using the techniques of quantum stochastic calculus, we construct dilations of the Bloch equations into boson and fermion 'noise baths'. Relaxation times and equilibrium values are computed in terms of the stochastic coupling parameters at different temperatures. The most general dilation of the standard form of the equations into a combination of boson and fermion noise is described using classical Brownian motion, the Clifford process and boson and fermion annihilation and creation processes. In an appendix, we extend the scheme to take account of Poisson processes.

## 1. Introduction

There has been much recent work (e.g. [1-7]), on the application of quantum stochastic calculus to the theory of dilations of norm continuous dynamical semigroups on von Neumann algebras. The physical motivation behind this is as follows: suppose that *So* is a quantised system undergoing an irreversible evolution through a quantised 'noise bath' $S_1$ (to which it is singularly coupled). We aim to extend the dynamics of *So* to a reversible dynamics of the combined systems $So+S_1$. Quantum stochastic calculus carries out this programme by providing a source of unitary cocycles for the free evolution on $S_1$ (at least in the case where $S_1$ may be described either exactly, or by means of an appropriate approximation, by a suitable Fock space [7]). The cocycles are obtained as the solutions of stochastic differential equations with respect to a quantum Brownian motion process [1, 3].

The quantum Brownian motion process is a family of pairs of annihilation and creation operators, together with a cyclic vector to determine expectations, acting on either boson or fermion Fock space. The annihilation and creation components of the process describe random absorption and emission (respectively) of quanta of $S_1$ by *So*. In the case where the rates of absorption and emission are equal, no overall quantum effects are detectable and the process reduces to a classical Brownian motion process ($S_1$ bosonic) or its anticommuting analogue, the Clifford process [8] ($S_1$ fermionic). We note that in this paper, the classical and Clifford processes always arise as momentum field observables.

Our aim in this paper is to use the techniques of quantum stochastic calculus to construct dilations of a particular case which we feel to be of some physical interest—namely when *So* is a two-level system described by the Bloch equations and $S_1$ is

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either bosonic, fermionic or, finally, a combination of both coupled independently to *So*.

The organisation of the paper is as follows. In § 2, we give a brief summary of the role of bosonic stochastic calculus in the construction of dilations. (We stress that our discussion is a limited one—a thorough exposition may be found in [4].) The gauge process is omitted from the main part of the paper and dealt with separately in an appendix. We include some account of stationarity in order to stress the different possibilities available in the finite temperature case and to make direct contact with the extensive dilation theory of Kümmerer and Schröder ([9-12] and references therein).

In §§ 3 and 4 we study the Bloch equations using bosonic stochastic calculus. An explicit computation of relaxation times and equilibrium values of the Pauli spin matrices is made, in terms of the parameters describing the stochastic coupling of *So* to $S_1$. We extensively study the equations in standard form [13] and classify all possible bosonic dilations in terms of the 'quantum diffusion equation' determined by the relevant unitary cocycle.

In §§ 5 and 6, we carry out the same programme for the case of fermionic $S_1$. It turns out that the fermion stochastic calculus developed in references [3, 14] is not quite general enough for our purposes, owing to unnatural parity assumptions on the stochastic integrals; we remedy this situation in § 5. We remark that the dilation schemes for the standard form in the bose and fermi cases are of identical structure; however, the stochastic processes associated with their quasi-free relaxations are markedly different. We suggest that this lack of symmetry is due to the fermionic nature of *So*.

We conclude, in § 7, by combining the analyses of §§ 4 and 6 to study the dilations of the standard form into a combined boson and fermion 'noise bath'. Note that from a physical viewpoint, no loss of generality is involved in treating the equations in standard form [13].

The main results of the paper are the following.

(i) The relaxation times and equilibrium values are both inversely proportional to the temperature of a bosonic 'noise bath'. In the fermionic case, the equilibrium values are again inversely proportional to the temperature but the relaxation times are constant.

(ii) The most general dilations of the Bloch equations in standard form into a combined bose and fermi 'noise bath' are given by the following two (mutually exclusive) possibilities. In either case, the quantum diffusion equation describing the dilation is generated by four independent processes, two of which are given by a classical Brownian motion process and a Clifford process. The other two are, in the first case, a pair of boson and fermion absorption (field annihilation) processes and in the second case, a pair of boson and fermion emission (field creation) processes.

Dilations of the Bloch equations, using different techniques, have been constructed in references [9, 10, 15]. (The dilation of [15] is clearly inequivalent to those discussed herein since it violates the Markov property [10].) Physical applicability of Markov dilations is discussed in [7]; for a criticism of this programme see [16].

We employ the following notation.

Let $h$ be a complex, separable Hilbert space and $\bar{h}$ be its dual. For $T$ a densely defined operator on $h$, we define the operator $\bar{T}$ on $\bar{h}$ by

$$\bar{T}\bar{f} = \overline{Tf} \text{ whenever } f \text{ lies in the domain of } T.$$

$T^\dagger$ will denote an operator on $h$, adjoint to $T$. $B(h)$ will denote the algebra of all bounded, linear operators on $h$.

For $S$, $T \in B(h)$, we denote the commutator

$$[S, T] = ST - TS$$

and the anticommutator

$$\{S, T\} = ST + TS.$$

For $U$ a unitary operator in $h$, we write

$$(\mathrm{Ad}\, U)X = UXU^\dagger \qquad (X \in B(h))$$

$\mathbb{R}^+ = [0, \infty)$.

## 2. Quantum stochastic calculus and dilations of dynamical semigroups

Let $h_0$ be a complex, separable Hilbert space and $\Gamma_B(L^2(\mathbb{R}))$ denote symmetric Fock space over $L^2(\mathbb{R})$. Let $\Omega_B = (1, 0, 0, \dots)$ be the vacuum vector in $\Gamma_B(L^2(\mathbb{R}))$ and for each $f, g \in L^2(\mathbb{R})$, let $a(f)$, $a^\dagger(g)$ denote the (boson) annihilation and creation operators (respectively) acting on $\Gamma_B(L^2(\mathbb{R}))$. For each $t \in \mathbb{R}^+$, write

$$A_t = I \otimes a(\chi_{[0,t)}), \qquad A_t^\dagger = I \otimes a^\dagger(\chi_{[0,t)})$$

these being mutually adjoint, densely defined operators on $h_B = h_0 \otimes \Gamma_B(L^2(\mathbb{R}))$.

Let $\omega_0$ be an arbitrary state on $B(h_0)$, $\omega_\Omega^B$ denote vacuum expectation on $B(\Gamma_B(L^2(\mathbb{R}))$ and $\omega$ be the state $\omega_0 \otimes \omega_\Omega^B$ on $B(h_B)$.

The family $\{(A_t, A_t^\dagger), t \in \mathbb{R}^+\}$ together with the state $\omega$ are a quantum Wiener process† of variance 1 in the sense of [17].

Let $L = L_0 \otimes I$ and $H = H_0 \otimes I$ be operators in $B(h_B)$ where $L_0$, $H_0 \in B(h_0)$ with $H_0 = H_0^\dagger$. It was shown in [1] that the quantum stochastic differential equation in $h_B$

$$\begin{aligned} \mathrm{d}U_t &= U_t[L\,\mathrm{d}A_t - L^\dagger\,\mathrm{d}A_t^\dagger + (\mathrm{i}H - \tfrac{1}{2}LL^\dagger)\,\mathrm{d}t] \\ U_0 &= I \end{aligned} \tag{2.1}$$

has a unique solution with each $U_t (t \in \mathbb{R}^+)$ a unitary operator on $h_B$.

The vacuum conditional expectation $\mathbb{E}_0^B : B(h_B) \to j(B(h_0))$ is given by continuous linear extension of the prescription

$$\begin{aligned} \mathbb{E}_0^B(X \otimes Y) &= j(X)\omega_\Omega^B(Y) \\ &= j(X)\langle \Omega_B, Y\Omega_B \rangle \end{aligned} \tag{2.2}$$

where $X \in B(h_0)$, $Y \in B(\Gamma_B(L^2(\mathbb{R}))$ and $j : B(h_0) \to B(h_B)$ is the canonical injection $j(X) = X \otimes I$.

It was found in [1] that the formula

$$\mathrm{P}^t(X) = j^{-1} \circ \mathbb{E}_0^B(U_t j(X) U_t^\dagger) \qquad \text{for } X \in B(h_0) \tag{2.3}$$

† Also called quantum Brownian motion [1].

defines a quantum dynamical semigroup $\mathrm{P}^t = \exp(t\mathscr{L})$ on $B(h_0)$, i.e. a one-parameter semigroup of norm continuous, identity preserving, completely positive maps of $B(h_0)$ into itself, with generator given by

$$\mathscr{L}(X) = \mathrm{i}[H_0, X] + L_0 X L_0^\dagger - \tfrac{1}{2}\{L_0 L_0^\dagger, X\} \tag{2.4}$$

for $X \in B(h_0)$ (cf [18]).

For each $t \in \mathbb{R}$, we denote by $S_t$ the shift in $L^2(\mathbb{R})$ defined by

$$(S_t f)(s) = f(t-s).$$

$\{S_t, t \in \mathbb{R}\}$ is a unitary group on $L^2(\mathbb{R})$, which lifts, through the functorial properties of second quantisation, to a unitary group $\{\Gamma_{\mathrm{B}}(S_t), t \in \mathbb{R}\}$ on $\Gamma_{\mathrm{B}}(L^2(\mathbb{R}))$.

From the cocycle property

$$U_{t+s} = U_s \alpha_s(U_t) \tag{2.5}$$

for $s, t \in \mathbb{R}^+$ established in [2] where $\alpha_t = \mathrm{Ad}(I \otimes \Gamma_{\mathrm{B}}(S_t))$ we see that $\{\hat{\mathrm{P}}^t, t \in \mathbb{R}\}$ is a group of automorphisms of $B(h_{\mathrm{B}})$ where for $Y \in B(h_{\mathrm{B}})$ [4, 6]

$$\begin{aligned} \hat{\mathrm{P}}^t(Y) &= \mathrm{Ad}\ U_t(\alpha_t(Y)) && \text{when } t \geq 0 \\ &= \alpha_t(\mathrm{Ad}\ U_{-t}^\dagger(Y)) && \text{when } t < 0. \end{aligned} \tag{2.6}$$

For $Y = j(X)$ with $X \in B(h_0)$ and $t \in \mathbb{R}^+$ we have

$$\begin{aligned} j^{-1} \circ \mathbb{E}_0^{\mathrm{B}}(\hat{\mathrm{P}}^t(Y)) &= j^{-1} \circ \mathbb{E}_0^{\mathrm{B}}(\mathrm{Ad}\ U_t(Y)) \\ &= \mathrm{P}^t(X) \end{aligned}$$

whence, for each $t \in \mathbb{R}^+$, the following diagram commutes:

$$\begin{array}{ccc} B(h_0) & \xrightarrow{\mathrm{P}^t} & B(h_0) \\ {\scriptstyle j}\big\downarrow & & \big\uparrow{\scriptstyle j^{-1} \circ \mathbb{E}_0^{\mathrm{B}}} \\ B(h_{\mathrm{B}}) & \xrightarrow{\hat{\mathrm{P}}^t} & B(h_{\mathrm{B}}) \end{array} .$$

We say that $(B(h_{\mathrm{B}}), \hat{\mathrm{P}}^t, j^{-1} \circ \mathbb{E}_0^{\mathrm{B}})$ is a bosonic stochastic dilation of $(B(h_0), \mathrm{P}^t)$.

We retain this terminology in the case where $h_{\mathrm{B}} = h_0 \otimes H$ where $H$ is isomorphic to symmetric Fock space over a direct sum (possibly infinite—see [2]) of copies of $L^2(\mathbb{R})$.

For $t \in \mathbb{R}$, we define a family of injections $j_t : B(h_0) \to B(h_{\mathrm{B}})$ by the prescription

$$j_t(X) = (\hat{\mathrm{P}}^t \circ j)X. \tag{2.7}$$

We note that the triple $(B(h_{\mathrm{B}}), \{j_t, t \in \mathbb{R}\}, \omega)$ is a quantum stochastic process in the sense of [19] and [7]. Furthermore, writing $X_t = j_t(X)$ for $t \in \mathbb{R}^+$, we obtain the stochastic differential equation [1]

$$\mathrm{d}X_t = [L_t, X_t]\,\mathrm{d}A_t - [L_t^\dagger, X_t]\,\mathrm{d}A_t^\dagger + \mathscr{L}(X_t)\,\mathrm{d}t \tag{2.8}$$

where $L_t = j_t(L_0)$ and $\mathscr{L}(X_t) = j_t(\mathscr{L}(X))(t \in \mathbb{R}^+)$.

An extension of the above theory has been developed by Hudson and Lindsay ([20-22]) using the quantum Wiener process $\{A_t^\phi, A_t^{\phi\dagger};\ t \in \mathbb{R}^+\}$ of variance $\sigma^2 = \cosh 2\phi$ ($\phi > 0$) in the state $\tilde{\omega} = \omega_0 \otimes \omega_\phi$ where $\omega_\phi$ is an extremal universally invariant quasifree state [23]. We realise the process in $\tilde{h}_{\mathrm{B}} = h_0 \otimes \Gamma_{\mathrm{B}}(L^2(\mathbb{R})) \otimes \Gamma_{\mathrm{B}}(\overline{L^2(\mathbb{R})})$ via the prescription

$$A_t^\phi = \cosh\phi(I \otimes a(\chi_{[0,t)}) \otimes I) + \sinh\phi(I \otimes I \otimes \bar{a}^\dagger(\overline{\chi_{[0,t)}}))$$

with $\omega_\phi$ acting as $\langle \Omega_{\mathrm{B}} \otimes \bar{\Omega}_{\mathrm{B}}, \cdot\ \Omega_{\mathrm{B}} \otimes \bar{\Omega}_{\mathrm{B}} \rangle$ where $\bar{\Omega}_{\mathrm{B}}$ is the vacuum vector in $\Gamma_{\mathrm{B}}(\overline{L^2(\mathbb{R})})$.

With appropriate modifications all of the theory discussed above carries over to this case. In particular $\mathbb{E}_0^{\mathrm{B}}$ is defined by substituting $\omega_\phi$ for $\omega_\Omega$ in (2.2), $\hat{\mathrm{P}}^t = \mathrm{Ad}(I\otimes\Gamma_{\mathrm{B}}(S_t)\otimes\Gamma_{\mathrm{B}}(\overline{S_t}))$ and (2.1) and (2.4) now take the forms

$$\mathrm{d}U_t = U_t(L\,\mathrm{d}A_t^{\phi} - L^\dagger\,\mathrm{d}A_t^{\phi\dagger} + (\mathrm{i}H - \tfrac{1}{2}\cosh^2\phi LL^\dagger - \tfrac{1}{2}\sinh^2\phi L^\dagger L)\mathrm{d}t) \tag{2.9}$$

$$\mathscr{L}(X) = \mathrm{i}[H_0, X] + \cosh^2\phi(L_0XL_0^\dagger - \tfrac{1}{2}\{L_0L_0^\dagger, X\}) + \sinh^2\phi(L_0^\dagger XL_0 - \tfrac{1}{2}\{L_0^\dagger L_0, X\}). \tag{2.10}$$

In (2.10) we may write, for $\beta > 0$

$$\cosh^2\phi = \frac{1}{1-\mathrm{e}^{-\beta}}, \qquad \sinh^2\phi = \frac{\mathrm{e}^{-\beta}}{1-\mathrm{e}^{-\beta}}$$

($\beta$ may be interpreted as an inverse temperature).

If there exists a faithful normal state $\omega_0$ on $B(h_0)$ whose associated modular automorphism group $\{\sigma_t;\, t\in\mathbb{R}\}$ satisfies

$$\sigma_t(H_0) = H_0 \qquad \sigma_t(L_0) = \exp(\mathrm{i}\beta\lambda t)L_0 \tag{2.11}$$

where $\lambda\in\mathbb{R}$, for all $t\in\mathbb{R}$, we deduce from (2.10) and (2.11), by theorem 4.2 of [24] that $\mathrm{P}^t$ satisfies the quantum detailed balance condition of [25] with respect to $\omega_0$ and furthermore that the state $\tilde{\omega} = \omega_0\otimes\omega_\phi$ is stationary on $B(h_{\mathrm{B}})$ in the sense that

$$\tilde{\omega}\circ\hat{\mathrm{P}}^t = \tilde{\omega}$$

whence, by (2.3), we find

$$\omega_0\circ\mathrm{P}^t = \omega_0.$$

We conclude that $(B(\tilde{h}_{\mathrm{B}}), \hat{\mathrm{P}}^t, \tilde{\omega}, j^{-1}\circ\mathbb{E}_0^{\mathrm{B}})$ is a stationary stochastic bosonic dilation of $(B(h_0), \mathrm{P}^t, \omega_0)$ as can be seen from the commutativity of the diagram

$$\begin{array}{ccc}
(B(h_0), \omega_0) & \xrightarrow{\mathrm{P}^t} & (B(h_0), \omega_0) \\
{\scriptstyle j}\big\downarrow & & \big\uparrow{\scriptstyle j^{-1}\circ\mathbb{E}_0^{\mathrm{B}}} \\
(B(\tilde{h}_{\mathrm{B}})\tilde{\omega}) & \xrightarrow{\hat{\mathrm{P}}^t} & (B(\tilde{h}_{\mathrm{B}}), \tilde{\omega}).
\end{array}$$

The general theory of stationary dilations on von Neumann algebras has been extensively studied in [9-12] (see also references therein). We will say that a bosonic stochastic dilation is of zero temperature if the cocycle $U_t$ satisfies (2.1) (i.e. $\beta=\infty$) and of finite temperature if $U_t$ satisfies (2.9) (i.e. $\beta<\infty$).

We remark that the most general form of the generator of a quantum dynamical semigroup is given by [18]

$$\mathscr{L}(X) = \sum_j (L_jXL_j^+ - \tfrac{1}{2}\{L_jL_j^+, X\} + \mathrm{i}[H, X]) \tag{2.12}$$

where the number of $L_j\in B(h_0)$ may be infinite, provided $\Sigma_j L_jL_j^+$ converges in the strong topology on $B(h_0)$. Dilations of such semigroups have been constructed in [2] by means of quantum stochastic calculus.

For the purposes of this paper, the greatest degree of generality we wish to consider is when the sum in (2.12) is finite, but each $L_j$ arises from a qualitatively different type of noise. Our results may then easily be extended to the most general form of (2.12) using similar techniques to those of [2].

## 3. Bosonic stochastic Bloch dilations and relaxation times

Fix $h_0 = \mathbb{C}^2$ so that $B(h_0) = M_2(\mathbb{C})$. We introduce the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\rho$ be a density matrix in $M_2(\mathbb{C})$. We define the polarisation components

$$M_x(t) = \mathrm{Tr}\, \rho \mathrm{P}^t(\sigma_x) \qquad M_y(t) = \mathrm{Tr}\, \rho \mathrm{P}^t(\sigma_y) \qquad M_z(t) = \mathrm{Tr}\, \rho \mathrm{P}^t(\sigma_z).$$

These are said to satisfy the Bloch equations [26] whenever there exist $\omega$, $\lambda_1$, $\lambda_2$, $\lambda_3$, $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3 \in \mathbb{R}$ such that

$$\begin{aligned} \mathrm{d}M_x(t)/\mathrm{d}t &= \omega M_y(t) - \lambda_1(M_x(t) - \varepsilon_1 I) \\ \mathrm{d}M_y(t)/\mathrm{d}t &= -\omega M_x(t) - \lambda_2(M_y(t) - \varepsilon_2 I) \\ \mathrm{d}M_z(t)/\mathrm{d}t &= -\lambda_3(M_z(t) - \varepsilon_3 I). \end{aligned} \tag{3.1}$$

We interpret $\omega$ as a Larmor frequency and $\lambda_j$, $\varepsilon_j$ $(j = 1, 2, 3)$ as inverse relaxation times and equilibrium values in the $x$, $y$ and $z$ directions respectively.

A bosonic stochastic dilation of $(M_2(\mathbb{C}), \mathrm{P}^t)$ satisfying (3.1) will be called a bosonic stochastic Bloch dilation. We restrict ourselves first to the zero temperature case.

We begin by considering the situation in which the equations are purely dissipative (i.e. $\omega = 0$). In this case a sufficient condition for (3.1) is given by

$$\mathscr{L}(\sigma_x) = -\lambda_1(\sigma_x - \varepsilon_1 I) \qquad \mathscr{L}(\sigma_y) = -\lambda_2(\sigma_y - \varepsilon_2 I) \qquad \mathscr{L}(\sigma_z) = -\lambda_3(\sigma_z - \varepsilon_3 I) \tag{3.2}$$

where $\mathscr{L}$ is of the form (2.4) with $H_0 = 0$.

Since $\mathscr{L}(I) = 0$, we may write (3.2) in the symbolic form

$$\mathscr{L}(\tau_j) = -\lambda_j \tau_j \qquad (j = 1, 2, 3) \tag{3.3}$$

where $\tau_1 = \sigma_x - \varepsilon_1 I$ etc.

We investigate the conditions under which (3.3) holds.

Let $\mathscr{R}_2 = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{C});\ a\bar{b} = c\bar{d},\ a\bar{c} = b\bar{d},\ a\bar{d} \in \mathbb{R},\ b\bar{c} \in \mathbb{R}\}$. We write $L_0 = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in M_2(\mathbb{C})$ in (2.4).

*Proposition 1.* A necessary and sufficient condition for (3.3) to hold is

$$L_0 \in \mathscr{R}_2. \tag{3.4}$$

Furthermore, in this case we obtain

$$\begin{aligned} \lambda_1 &= \tfrac{1}{2}(|\alpha - \delta|^2 + |\beta - \gamma|^2) \\ \lambda_2 &= \tfrac{1}{2}(|\alpha - \delta|^2 + |\beta + \gamma|^2) \\ \lambda_3 &= |\beta|^2 + |\gamma|^2 \end{aligned} \tag{3.5}$$

$$\begin{aligned} \varepsilon_1 &= (2/\lambda_1)(\mathrm{Re}\ \alpha\bar{\gamma} - \mathrm{Re}\ \alpha\bar{\beta}) \\ \varepsilon_2 &= (2/\lambda_2)(\mathrm{Im}\ \alpha\bar{\gamma} + \mathrm{Im}\ \alpha\bar{\beta}) \\ \varepsilon_3 &= (1/\lambda_3)(|\gamma|^2 - |\beta|^2). \end{aligned} \tag{3.6}$$

The proof is by straightforward computation.

*Remark.* Let $\mathscr{R}_2(\mathbb{R}) = \mathscr{R}_2 \cap M_2(\mathbb{R})$. It is easy to see that $\mathscr{R}_2(\mathbb{R})$ is a semigroup under matrix multiplication. Furthermore, the set of non-singular elements of $\mathscr{R}_2(\mathbb{R})$ are a closed subgroup of $\mathrm{GL}(2, \mathbb{R})$.

Let $\Sigma_3$ denote the symmetric group on $\{1, 2, 3\}$. From (2.5) we obtain, for every $g \in \Sigma_3$ [27]

$$\lambda_{g(1)} + \lambda_{g(2)} \geqslant \lambda_{g(3)}. \tag{3.7}$$

We return now to the most general form of (3.1) with $\omega \neq 0$. Let us suppose that we have, generalising (3.2)

$$\begin{aligned} \mathscr{L}(\sigma_x) &= \omega\sigma_y - \lambda_1(\sigma_x - \varepsilon_1 I) \\ \mathscr{L}(\sigma_y) &= -\omega\sigma_x - \lambda_2(\sigma_y - \varepsilon_2 I) \\ \mathscr{L}(\sigma_z) &= -\lambda_3(\sigma_z - \varepsilon_3 I) \end{aligned} \tag{3.8}$$

where we have taken $H_0 = \frac{1}{2}\omega\sigma_z$ in (2.4).

Equations (3.8) are a sufficient condition for (3.1) if and only if the dissipative and Hamiltonian parts of $\mathscr{L}$ commute. From (3.8) we see that this is so if and only if

$$\lambda_1 = \lambda_2 \qquad \text{and} \qquad \varepsilon_1 = \varepsilon_2 = 0.$$

When this is the case, we say that the Bloch equations (3.1) and any bosonic stochastic dilation of the correspondent semigroup are in standard form [13]. We will investigate this case in more detail in the next section.

We conclude this section by generalising proposition 1 to the finite temperature case. Taking $\mathscr{L}$ as in (2.10) (with $H_0 = 0$) we obtain once again the condition (3.4) and the following values for the parameters:

$$\begin{aligned} \lambda_1^\phi &= \tfrac{1}{2}\sigma^2(|\alpha - \delta|^2 + |\beta - \gamma|^2) \\ \lambda_2^\phi &= \tfrac{1}{2}\sigma^2(|\alpha - \delta|^2 + |\beta + \gamma|^2) \\ \lambda_3^\phi &= \sigma^2(\gamma^2 + \beta^2) \end{aligned} \tag{3.9}$$

$$\begin{aligned} \varepsilon_1^\phi &= (2/\lambda_1^\phi)(\mathrm{Re}\, \alpha\bar{\gamma} - \mathrm{Re}\, \alpha\bar{\beta}) \\ \varepsilon_2^\phi &= (2/\lambda_2^\phi)(\mathrm{Im}\, \alpha\bar{\beta} + \mathrm{Im}\, \alpha\bar{\gamma}) \\ \varepsilon_3^\phi &= (1/\lambda_3^\phi)(|\gamma|^2 - |\beta|^2). \end{aligned} \tag{3.10}$$

Comparing (3.5) and (3.9), we obtain

$$\lambda_j^\phi = \sigma^2 \lambda_j \tag{3.11}$$

and (3.6) and (3.10) yield

$$\varepsilon_j^\phi = (1/\sigma^2)\varepsilon_j. \tag{3.12}$$

Now the variance

$$\begin{aligned} \sigma^2 &= \cosh^2\phi + \sinh^2\phi \\ &= (1 + \mathrm{e}^{-\beta})/(1 - \mathrm{e}^{-\beta}) \\ &= \coth \tfrac{1}{2}\beta \propto \beta^{-1} \end{aligned}$$

whence we conclude from (3.11) and (3.12) that the relaxation times and equilibrium values are both inversely proportional to the temperature of the dilation.

## 4. Bosonic stochastic Bloch dilations in standard form

Throughout this section, we will for simplicity restrict our analysis to the zero temperature case. It is easily verified that this involves no significant loss in generality.

The Bloch equations (3.1) in standard form are

$$\begin{aligned} \mathrm{d}M_x(t)/\mathrm{d}t &= \omega M_y(t) - \lambda M_x(t) \\ \mathrm{d}M_y(t)/\mathrm{d}t &= -\omega M_x(t) - \lambda M_y(t) \\ \mathrm{d}M_z(t)/\mathrm{d}t &= -\mu(M_z(t) - \varepsilon I) \end{aligned} \tag{4.1}$$

where we have written $\lambda = \lambda_1 = \lambda_2$, $\mu = \lambda_3$ and $\varepsilon = \varepsilon_3$.

We see from (3.5) that

$$\lambda_1 = \lambda_2 \qquad \text{if and only if } \beta = 0 \text{ or } \gamma = 0. \tag{4.2}$$

Let us, without loss of generality, take $\gamma = 0$. We introduce the transverse relaxation time

$$T_\perp = \lambda^{-1} = 2(|\beta|^2 + |\alpha - \delta|^2)^{-1}$$

and the longitudinal relaxation time

$$T_\parallel = \mu^{-1} = |\beta|^{-2}.$$

These clearly satisfy the relation [28]

$$T_\parallel \geqslant \tfrac{1}{2} T_\perp. \tag{4.3}$$

It is remarked in [27] that (4.3) is always satisfied experimentally.

We will now make an explicit study of bosonic stochastic dilations of (4.1). Any $L_0 \in M_2(\mathbb{C})$ may be written

$$L_0 = \alpha a a^\dagger + \beta a + \gamma a^\dagger + \delta a^\dagger a$$

where $\alpha$, $\beta$, $\gamma$, $\delta \in \mathbb{C}$,

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad a^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We note that

$$a^2 = (a^\dagger)^2 = 0 \qquad \text{and} \qquad \{a, a^\dagger\} = I.$$

Using the notation of § 2, we write $a_t = j_t(a)$ and explicitly compute the form of (2.8) to obtain

$$\mathrm{d}a_t = \{(\alpha - \delta)a_t - \gamma[a_t, a_t^\dagger]\}\,\mathrm{d}A_t - \{(\bar{\alpha} - \bar{\delta})a_t - \bar{\beta}[a_t, a_t^\dagger]\}\,\mathrm{d}A_t^\dagger + \mathscr{L}(a_t)\,\mathrm{d}t. \tag{4.4}$$

We have not written out the $\mathrm{d}t$ term fully since it plays no role in our subsequent analysis. We remark that (4.4) defines a quantum diffusion process of similar type to those investigated in [29].

Now let $\phi = \arg(\alpha - \delta)$, $\beta' = \mathrm{e}^{-\mathrm{i}\phi}\beta$ and $\gamma' = \mathrm{e}^{-\mathrm{i}\phi}\gamma$. Since the quantum Wiener process is invariant under the action of the gauge group U(1) [26] given by

$$A_t \to \mathrm{e}^{\mathrm{i}\phi} A_t^\dagger, \qquad A_t^\dagger \to \mathrm{e}^{-\mathrm{i}\phi} A_t^\dagger$$

we may transform (4.4) into the equation

$$\mathrm{d}a_t = \mathrm{i}|\alpha - \delta| a_t\,\mathrm{d}P_t - \gamma'[a_t, a_t^\dagger]\,\mathrm{d}A_t + \bar{\beta}'[a_t, a_t^\dagger]\,\mathrm{d}A_t^\dagger + \mathscr{L}(a_t)\,\mathrm{d}t \tag{4.5}$$

where $P_t = -\mathrm{i}(A_t - A_t^\dagger)$ $(t \in \mathbb{R}^+)$.

Let $\mathscr{C} = \{f : \mathbb{R} \to \mathbb{R}, f(0) = 0 \text{ and } f \text{ continuous}\}$ and let $\mu$ denote the Wiener measure on $\mathscr{C}$. We realise the classical Brownian motion process $\{X_t, t \in \mathbb{R}^+\}$ on $\mathscr{C}$ by

$$X_t(f) = f(t) \qquad (t \in \mathbb{R}^+).$$

There is a unique Hilbert space isomorphism $D$ (called the duality transform [30]) from $\Gamma_B(L^2(\mathbb{R}))$ to $L^2(\mathscr{C}, \mu)$ such that for each $t \in \mathbb{R}^+$,

$$DP_tD^{-1} = X_t. \tag{4.6}$$

Now, by (4.2) and (4.4) there are three possible forms for (4.5) when the dilation is in standard form, i.e.

$$\mathrm{d}a_t = -\gamma'[a_t, a_t^\dagger]\,\mathrm{d}A_t + \mathscr{L}(a_t)\,\mathrm{d}t \tag{4.7a}$$

$$\mathrm{d}a_t = +\bar{\beta}'[a_t, a_t^\dagger]\,\mathrm{d}A_t^\dagger + \mathscr{L}(a_t)\,\mathrm{d}t \tag{4.7b}$$

$$\mathrm{d}a_t = \mathrm{i}|\alpha - \delta|a_t\,\mathrm{d}P_t + \mathscr{L}(a_t)\,\mathrm{d}t. \tag{4.7c}$$

We consider each of these cases in turn.

(i) $$\mathrm{d}a_t = -\gamma'[a_t, a_t^\dagger]\,\mathrm{d}A_t + (\mathrm{i}\omega a_t - \tfrac{1}{2}|\gamma|^2 a_t)\,\mathrm{d}t. \tag{4.8}$$

In this case the cocycle for the group dilation satisfies the equation

$$\mathrm{d}U_t^\uparrow = U_t^\uparrow j[(\gamma' a^\dagger\,\mathrm{d}A_t - \gamma' a\,\mathrm{d}A_t^\dagger) - \tfrac{1}{2}(|\gamma'|^2 a^\dagger a - \mathrm{i}\omega[a, a^\dagger])\,\mathrm{d}t] \tag{4.9}$$

an explicit solution of which may be calculated using the technique of [6].

We compute the semigroup $\mathrm{P}_\uparrow^t = \mathrm{e}^{t\mathscr{L}_\uparrow}$ dilated by (4.9) and obtain for

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathbb{C})$$

$$\mathrm{P}_\uparrow^t(A) = \begin{pmatrix} a_{11} & \exp[-(\frac{1}{2}|\gamma|^2 - \mathrm{i}\omega)t]a_{12} \\ \exp[-(\frac{1}{2}|\gamma|^2 + \mathrm{i}\omega)t]a_{21} & a_{22} + [1 - \exp(-|\gamma|^2 t)](a_{11} - a_{22}) \end{pmatrix}. \tag{4.10}$$

Putting $\omega = 0$ in (4.10) yields an example of the quasi-free relaxation introduced in [9].

(ii) This is very similar to (i), with equations (4.8)-(4.10) replaced by

$$\mathrm{d}a_t = \bar{\beta}'[a_t, a_t^\dagger]\,\mathrm{d}A_t^\dagger + (\mathrm{i}\omega a_t - \tfrac{1}{2}|\beta'|^2 a_t)\,\mathrm{d}t \tag{4.11}$$

$$\mathrm{d}U_t^\downarrow = U_t^\downarrow j[(\beta' a\,\mathrm{d}A_t - \bar{\beta}' a^\dagger\,\mathrm{d}A_t^\dagger - \tfrac{1}{2}(|\beta'|^2 a a^\dagger - \mathrm{i}\omega[a, a^\dagger])\,\mathrm{d}t] \tag{4.12}$$

$$\mathrm{P}_\downarrow^t(A) = \begin{pmatrix} a_{11} + [1 - \exp(-|\beta|^2 t)](a_{22} - a_{11}) & \exp[-(\frac{1}{2}|\beta|^2 - \mathrm{i}\omega)t]a_{12} \\ \exp[-(\frac{1}{2}|\beta|^2 + \mathrm{i}\omega)t]a_{21} & a_{22} \end{pmatrix} \tag{4.13}$$

respectively; (4.12) is explicitly solved in [6]. With $\omega = 0$ in (4.13) we again obtain a quasi-free relaxation.

(iii) By virtue of (4.6) we can consider this case as an equation in $h_0 \otimes L^2(\mathscr{C}, \mu)$

$$\mathrm{d}a_t = \mathrm{i}|\alpha - \delta|a_t\,\mathrm{d}X_t + (\mathrm{i}\omega - \tfrac{1}{2}|\alpha - \delta|^2)a_t\,\mathrm{d}t \tag{4.14}$$

the solution of which is

$$a_t = j(a)\exp(\mathrm{i}|\alpha - \delta|X_t + \mathrm{i}\omega t) \tag{4.15}$$

and a cocycle for the dilation is

$$U_t^c = \exp[\mathrm{i}j([a, a^\dagger])(\tfrac{1}{2}|\alpha - \delta|X_t + \omega t)]. \tag{4.16}$$

(We remark that this is a special case of a structure developed in [31].)

The semigroup is

$$\mathrm{P}_c^t(A)=\begin{pmatrix} a_{11} & \exp[-(\frac{1}{2}|\alpha-\delta|^2-\mathrm{i}\omega)t]a_{12} \\ \exp[-(\frac{1}{2}|\alpha-\delta|^2+\mathrm{i}\omega)t]a_{21} & a_{22}\end{pmatrix} \tag{4.17}$$

which is an example of the Larmor relaxation of [9] (see also [12], particularly the theorem on p 237).

From (4.10), (4.13) and (4.17) we see that the semigroups satisfy the following commutation relations:

$$[\mathrm{P}_c^t, \mathrm{P}_\uparrow^t]=[\mathrm{P}_c^t, \mathrm{P}_\downarrow^t]=0 \qquad [\mathrm{P}_\uparrow^t, \mathrm{P}_\downarrow^t]\neq 0 \qquad \forall t\in\mathbb{R}^+. \tag{4.18}$$

We introduce the dynamical semigroups

$$\mathrm{P}_{c\uparrow}^t=\mathrm{P}_c^t\circ\mathrm{P}_\uparrow^t \qquad \mathrm{P}_{c\downarrow}^t=\mathrm{P}_c^t\circ\mathrm{P}_\downarrow^t. \tag{4.19}$$

Let $\mathcal{N}_{\mathrm{B}}=\mathbb{C}^2\otimes L^2(\mathscr{C},\mu)\otimes\Gamma_{\mathrm{B}}(L^2(\mathbb{R}))$ and $j$ be the canonical injection of $M_2(\mathbb{C})$ into $B(\mathcal{N}_{\mathrm{B}})$. We denote by $\mathbb{E}_0^{\mathrm{c}}$ the conditional expectation on $\mathbb{C}^2\otimes L^2(\mathscr{C},\mu)$ with respect to Wiener measure and $\{\mathscr{S}_t, t\in\mathbb{R}\}$ the unitary group of shift operators in $L^2(\mathscr{C},\mu)$ i.e.

$$\mathscr{S}_t=D\Gamma(S_t)D^{-1} \qquad \forall t\in\mathbb{R}.$$

We define

$$\begin{aligned}\hat{\mathrm{P}}_t^c&=\mathrm{Ad}(U_t^c\mathscr{S}_t), & t&\geq 0\\ &=\mathrm{Ad}(\mathscr{S}_tU_{-t}^{c\dagger}), & t&<0\end{aligned}$$

whence we see that $(B(\mathbb{C}^2\otimes L^2(\mathscr{C},\mu)), \hat{\mathrm{P}}_t^c, j^{-1}\circ\mathbb{E}_0^{\mathrm{c}})$ is a bosonic dilation of $(B(h_0), \mathrm{P}_c^t)$, equivalent via $D$ to the bosonic dilation constructed in (iii) with $\beta=\gamma=0$.

We write $\hat{\mathrm{P}}_\uparrow^t, \hat{\mathrm{P}}_\downarrow^t$ $(t\in\mathbb{R})$, for the groups implementing the dilations of $\mathrm{P}_\uparrow^t, \mathrm{P}_\downarrow^t$ $(t\in\mathbb{R}^+)$ respectively discussed in (i) and (ii) above.

Our main result of this section is the following theorem.

*Theorem 2.* $(B(\mathcal{N}_{\mathrm{B}}), \hat{\mathrm{P}}_c^t\circ\hat{\mathrm{P}}_\downarrow^t, j^{-1}\circ\mathbb{E}_0^{\mathrm{B}}\circ\mathbb{E}_0^{\mathrm{c}})$ and $(B(\mathcal{N}_{\mathrm{B}}), \hat{\mathrm{P}}_c^t\circ\hat{\mathrm{P}}_\uparrow^t, j^{-1}\circ\mathbb{E}_0^{\mathrm{B}}\circ\mathbb{E}_0^{\mathrm{c}})$ are bosonic stochastic dilations of $(M_2(\mathbb{C}), \mathrm{P}_{c\downarrow}^t)$ and $(M_2(\mathbb{C}), \mathrm{P}_{c\uparrow}^t)$, respectively.

The proof is trivial. (In the statement of theorem 2, for notational convenience, we have omitted the canonical injections which extend $\mathbb{E}_0^{\mathrm{B}}$, $\mathbb{E}_0^{\mathrm{c}}$, $\hat{\mathrm{P}}_c^t$, $\hat{\mathrm{P}}_\downarrow^t$ and $\mathrm{P}_\uparrow^t$ onto the whole of $B(\mathcal{N}_{\mathrm{B}})$.)

In either of the two cases of theorem 2, the underlying quantum stochastic process satisfies the corresponding stochastic differential equation (SDE) of (4.7). Thus we see that theorem 2 provides the most general description of the standard Bloch equations for a (zero temperature) bosonic dilation via quantum Brownian motion.

The form of (4.7) suggests the following physical interpretation: that we regard the standard bosonic dilation as arising from the combination of a purely classical process (e.g. a random magnetic field) and a purely quantum process (e.g. a bosonic radiation field) interacting with the two-level system either by absorption or emission.

In [27], the standard form was characterised in terms of rotational symmetry about the $\sigma_z$ direction. We conclude this section by examining this result within the context of our dilation theory.

Let $\mathrm{G}=\{\mathrm{e}^{\mathrm{i}\theta\sigma_z}, \theta\in[0,2\pi)\}$. G is a one-parameter subgroup of SU(2) which has a unitary representation on $\mathbb{C}^2$, acting on $M_2(\mathbb{C})$ by algebraic extension of

$$(\mathrm{Ad}\ V_\theta)a=\mathrm{e}^{\mathrm{i}\theta}a \qquad (\mathrm{Ad}\ V_\theta)a^\dagger=\mathrm{e}^{-\mathrm{i}\theta}a^\dagger$$

for each $V_\theta=\mathrm{e}^{\mathrm{i}\theta\sigma_z}\in\mathrm{G}$.

*Theorem 3.* The following are equivalent, for fixed $H_0=\frac{1}{2}\omega\sigma_z$ in (2.4) and for all $\theta\in[0,2\pi]$.

(i) For each $X\in M_2(\mathbb{C})$, $t\in\mathbb{R}^+$

$$\mathrm{Ad}\ V_\theta \mathrm{P}^t(X)=\mathrm{P}^t(\mathrm{Ad}\ V_\theta X).$$

(ii) $\mathrm{Ad}\ V_\theta(L_0)=L_0$.

(iii) $\mathrm{Ad}\, j(V_\theta)(U_t)=U_t$, $\forall t\in\mathbb{R}^+$.

(iv) For each $Y\in B(h_\mathrm{B})$, $t\in\mathbb{R}$

$$\mathrm{Ad}\, j(V_\theta)\hat{\mathrm{P}}^t(Y)=\hat{\mathrm{P}}^t(\mathrm{Ad}\, j(V_\theta)Y).$$

(v) The Bloch equations are in standard form.

*Proof.* (i) is equivalent to

$$\mathrm{Ad}\ V_\theta\, \mathrm{e}^{t\mathscr{L}}(X)=\mathrm{e}^{t\mathscr{L}}(\mathrm{Ad}\ V_\theta X)\Leftrightarrow \mathrm{Ad}\ V_\theta\mathscr{L}(X)=\mathscr{L}(\mathrm{Ad}\ V_\theta X)$$

$$\Leftrightarrow\ V_\theta(L_0XL_0^\dagger-\tfrac{1}{2}L_0L_0^\dagger X-\tfrac{1}{2}XL_0L_0^\dagger+\mathrm{i}[H_0,X])V_\theta^\dagger \tag{a}$$

$$=L_0V_\theta XV_\theta^\dagger L_0^\dagger-\tfrac{1}{2}L_0L_0^\dagger V_\theta XV_\theta^\dagger-\tfrac{1}{2}V_\theta XV_\theta^\dagger L_0L_0^\dagger+\mathrm{i}[H_0,V_\theta XV_\theta^\dagger].$$

However, $V_\theta H_0V_\theta^\dagger=H_0\ (\theta\in[0,2\pi])$ whence $V_\theta[H_0,X]V_\theta^\dagger=[H_0,V_\theta XV_\theta^\dagger]$ so (a)$\Leftrightarrow$

$$L_0=V_\theta L_0V_\theta^\dagger$$

$\Leftrightarrow$ the cocycle for the dilation is given by the solution of

$$\mathrm{d}U_t=U_t(j(V_\theta)Lj(V_\theta^\dagger)\,\mathrm{d}A_t-j(V_\theta)L^+j(V_\theta^\dagger)\,\mathrm{d}A^+$$
$$+(\mathrm{i}j(V_\theta)Hj(V_\theta^\dagger)-\tfrac{1}{2}j(V_\theta)LL^+j(V_\theta^\dagger))\,\mathrm{d}t)\qquad \text{with } U_0=I$$

$$\Leftrightarrow \mathrm{d}(U_tj(V_\theta))=U_tj(V_\theta)(L\,\mathrm{d}A_t-L^+\,\mathrm{d}A_t^\dagger+(\mathrm{i}H-\tfrac{1}{2}LL^+)\,\mathrm{d}t\qquad \text{with } U_0=I$$

$$\Leftrightarrow \mathrm{d}(j(V_\theta)U_tj(V_\theta^\dagger))=j(V_\theta)U_tj(V_\theta^\dagger)(L\,\mathrm{d}A_t-L^+\,\mathrm{d}A_t^\dagger$$
$$+(\mathrm{i}H-\tfrac{1}{2}LL^+\,\mathrm{d}t))\qquad \text{with } j(V_\theta)U_0j(V_\theta^\dagger)=I$$

whence by the uniqueness theorem for solutions to equation (2.1) [1] we conclude

$$j(V_\theta)U_tj(V_\theta^\dagger)=U_t\ \forall t\in\mathbb{R}^+\Leftrightarrow\forall t\in\mathbb{R}^+,\ Y\in B(h_\mathrm{B})$$

$$\mathrm{Ad}\, j(V_\theta)\hat{\mathrm{P}}^t(Y)$$
$$=j(V_\theta)U_t\alpha_t(Y)U_t^\dagger j(V_\theta^\dagger)\qquad \text{by (2.6)}$$
$$=U_tj(V_\theta)\alpha_t(Y)j(V_\theta^\dagger)U_t^\dagger$$
$$=U_t\alpha_t(\mathrm{Ad}\, j(V_\theta)Y)U_t^+$$
$$=\hat{\mathrm{P}}^t(\mathrm{Ad}\, j(V_\theta)Y).$$

The corresponding statement for $t<0$ follows by adjunction in (iii).

(v) is true if and only if (4.5) takes the forms

$$\mathrm{d}a_t=\mathrm{i}|\alpha-\delta|a_t\,\mathrm{d}P_t-\mathrm{e}^{-\mathrm{i}\theta}\gamma'[a_t,a_t^\dagger]\,\mathrm{d}A_t+\mathrm{e}^{-\mathrm{i}\theta}\bar{\beta}'[a_t,a_t^\dagger]\,\mathrm{d}A_t^\dagger+\mathscr{L}(a_t)\,\mathrm{d}t \tag{b}$$

and

$$\mathrm{d}a_t=\mathrm{i}|\alpha-\delta|a_t\,\mathrm{d}P_t-\gamma'[a_t,a_t^\dagger]\,\mathrm{d}A_t+\bar{\beta}'[a_t,a_t^\dagger]\,\mathrm{d}A_t^\dagger+\mathscr{L}(a_t)\,\mathrm{d}t. \tag{c}$$

In (b) we can absorb one of the $\mathrm{e}^{-\mathrm{i}\theta}$ factors by making either of the gauge transformations $A_t\to\mathrm{e}^{-\mathrm{i}\theta}A_t$, $A_t\to\mathrm{e}^{\mathrm{i}\theta}A_t$. In the first case (b) and (c) are equal$\Leftrightarrow\beta=0$ and in the second case $\Leftrightarrow\gamma=0$ (the only other possibility for equality is $\beta=\gamma=0$).

Let $\mathscr{U}_\theta = \text{Ad } V_\theta$ and $\mathscr{U}^j_\theta = \text{Ad}\, j(V_\theta)$ $(\theta \in [0, 2\pi))$. Theorem 3, for the given choice of $H$, demonstrates that the Bloch equations are in standard form if and only if G induces a one-parameter group of symmetries of the dilation in the sense that all $\theta \in [0, 2\pi)$.

$(B(h_B), (\mathscr{U}^j_\theta)^{-1}\hat{\text{P}}'(\mathscr{U}^j_\theta), j^{-1} \circ \mathbb{E}^B_0)$ is a bosonic stochastic dilation of $(M_2(\mathbb{C}), \mathscr{U}_\theta^{-1}\text{P}'\mathscr{U}_\theta)$.

## 5. Fermionic stochastic dilations

We recall the following definitions and notation from [32] and [3].

A complex separable Hilbert space $\mathscr{H}$ is said to be $Z_2$-graded if it may be written

$$\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-.$$

We refer to $\mathscr{H}_+$ and $\mathscr{H}_-$ as the even and odd subspaces (respectively). $T \in B(\mathscr{H})$ is said to be even if $T\mathscr{H}_\pm \subset \mathscr{H}_\pm$ and odd if $T\mathscr{H}_\pm \subset \mathscr{H}_\mp$ whence we see that $B(\mathscr{H})$ is a $Z_2$-graded algebra in the sense of [33].

We denote by $\rho$ the parity automorphism of $B(\mathscr{H})$ for which

$$\rho(T) = T \qquad \text{if } T \text{ is even}$$

$$\rho(T) = -T \qquad \text{if } T \text{ is odd.}$$

$\rho$ is unitarily implementable by a self-adjoint unitary operator $\theta$ on $\mathscr{H}$ satisfying $\theta^2 = I$.

Let $\mathscr{H}_1$, $\mathscr{H}_2$ be $Z_2$-graded Hilbert spaces and $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$. Clearly $\mathscr{H}$ is $Z_2$-graded. Let $\psi_i \in \mathscr{H}_i$ and $T_i \in B(\mathscr{H}_i)$ $(i = 1, 2)$ with $\psi_1$ and $T_2$ of definite parity.

The Chevalley product $T_1 \hat{\otimes} T_2$ is defined by continuous linear extension of

$$(T_1 \hat{\otimes} T_2)(\psi_1 \otimes \psi_2) = (-1)^{\delta(T_2)\varepsilon(\psi_1)} T_1\psi_1 \otimes T_2\psi_2 \tag{5.1}$$

where $\delta(T_2) = \text{sgn } \rho(T_2)$ and $\varepsilon(\psi_1) = \text{sgn } \theta\psi_1$. Equation (5.1) extends by linearity to the case where $T_2$ is an arbitrary element of $B(\mathscr{H}_2)$.

For $S_i$, $T_i \in B(\mathscr{H}_i)$ $(i = 1, 2)$ with $S_2$ and $T_1$ of definite parity, we have

$$(S_1 \hat{\otimes} S_2)(T_1 \hat{\otimes} T_2) = (-1)^{\delta(S_2)\delta(T_1)} S_1T_1 \hat{\otimes} S_2T_2 \tag{5.2}$$

which, again, extends by linearity to the case of arbitrary $S_2$, $T_1$.

Let $\Gamma_F(L^2(\mathbb{R}))$ denote antisymmetric Fock space over $L^2(\mathbb{R})$. It is $Z_2$-graded by means of its decomposition into direct sums of odd and even antisymmetric tensor powers. For $f$, $g \in L^2(\mathbb{R})$, let $b(f)$ and $b^\dagger(g)$ denote the (fermion) annihilation and creation operators (respectively) acting on $\Gamma_F(L^2(\mathbb{R}))$. These are clearly odd operators. We write $\rho' = \text{Ad } \theta'$ for the parity automorphism of $B(\Gamma_F(L^2(\mathbb{R})))$ and $\Omega_F = (1, 0, 0, \ldots)$ for the vacuum vector in $\Gamma_F(L^2(\mathbb{R}))$.

Let $h_0$ be a complex, separable $Z_2$-graded Hilbert space and let $\rho_0 = \text{Ad } \theta_0$ denote the parity automorphism of $B(h_0)$. We write $h_F = h_0 \otimes \Gamma_F(L^2(\mathbb{R}))$. Clearly the parity automorphism $\rho = \text{Ad } \theta$ of $B(h_F)$ is given by $\rho = \rho_0 \otimes \rho'$.

Let $\iota : B(h_0) \to B(h_F)$ denote the canonical injection given by $\iota(A) = A \hat{\otimes} I$ $(A \in B(h_0))$. The vacuum conditional expectation $\mathbb{E}^F_0 : B(h_F) \to \iota(B(h_0))$ is defined as in (2.2) with $\iota$ and $\Omega_F$ replacing $j$ and $\Omega_B$ (respectively).

Let $B_t = I \hat{\otimes} b(\chi_{[0,t)})$, $B^\dagger_t = I \hat{\otimes} b^\dagger(\chi_{[0,t)})$ $(t \in \mathbb{R}^+)$ and $\omega' = \omega_0 \otimes \omega^F_\Omega$ where $\omega_0$ is an arbitrary state on $B(h_0)$ and $\omega^F_\Omega$ is vacuum expectation on $\Gamma_F(L^2(\mathbb{R}))$. The family $\{(B_t, B^\dagger_t);\ t \in \mathbb{R}^+\}$ together with the state $\omega'$ yields a fermion Brownian motion process of variance 1 in the sense of [34].

Fermion analogues of (2.1) and (2.3) were developed in [3]. However the dilation theory so obtained was limited by the proviso that $L_0$ and $H_0$ be odd and even elements of $B(h_0)$, respectively; consequently only even cocycles were admissible.

The following sequence of propositions provides the relevant generalisation of theorems 5.2, 6.3 and 7.1(*b*) of [3]. In each case, the proof is a straightforward modification.

We refer the reader to [3] for the definitions of adapted process, locally square integrable process and stochastic integral.

Let $\mathscr{A}$ denote the set of adapted processes $\{M_t, t\in\mathbb{R}^+\}$ in $h_F$ satisfying

$$\mathrm{d}M_t=\mathrm{d}B_t^\dagger F_t+G_t\,\mathrm{d}B_t+H_t\,\mathrm{d}t$$

with $\{F_t, t\in\mathbb{R}^+\}$, $\{G_t, t\in\mathbb{R}^+\}$ and $\{H_t, t\in\mathbb{R}^+\}$ locally square integrable processes such that

$$M_t, F_t, G_t, H_t\in B(h_F)$$

and

$$\sup_{0\leqslant s\leqslant t}\max\{\|M_s\|, \|F_s\|, \|G_s\|, \|H_s\|\}<\infty$$

for each $t\in\mathbb{R}^+$.

*Proposition 4.* $\mathscr{A}$ is a *-algebra under pointwise operator multiplication and the involution $M_t\to M_t^\dagger(t\in\mathbb{R}^+)$. Furthermore, for $\{M^i(t), t\in\mathbb{R}^+\}\in\mathscr{A}$ $(i=1,2)$ with each

$$\mathrm{d}M_t^i=\mathrm{d}B_t^\dagger F_t^i+G_t^i\,\mathrm{d}B_t+H_t^i\,\mathrm{d}t$$

we have

$$\mathrm{d}(M_t^1M_t^2)=\mathrm{d}M_t^1M_t^2+M_t^1\,\mathrm{d}M_t^2+\mathrm{d}M_t^1\,\mathrm{d}M_t^2$$

where

$$\mathrm{d}M_t^1M_t^2=\mathrm{d}B_t^\dagger F_t^1M_t^2+G_t^1\rho(M_t^2)\,\mathrm{d}B_t+H_t^1M_t^2\,\mathrm{d}t$$
$$M_t^1\,\mathrm{d}M_t^2=\mathrm{d}B_t^\dagger\rho(M_t^1)F_t^2+M_t^1G_t^2\,\mathrm{d}B_t+M_t^1H_t^2\,\mathrm{d}t$$
$$\mathrm{d}M_t^1\,\mathrm{d}M_t^2=G_t^1F_t^2\,\mathrm{d}t.$$

Proposition 1 is Itô's formula in $h_F$. It generalises theorem 4.2 of [3] to the extent that $\{M_t^i, t\in\mathbb{R}^+\}$, $i=1, 2$, are no longer required to be of definite parity.

*Proposition 5.* For arbitrary $L_0, H_0\in B(h_0)$, with $H_0=H_0^\dagger$, there exists a unique solution to the stochastic differential equation

$$\begin{aligned}\mathrm{d}V_t&=V_t(\iota(L_0)\,\mathrm{d}B_t-\rho(\iota(L_0^\dagger))\,\mathrm{d}B_t^\dagger)+[\mathrm{i}\iota(H_0)-\tfrac{1}{2}\iota(L_0L_0^\dagger)]\,\mathrm{d}t)\\V_0&=I\end{aligned}\tag{5.3}$$

such that each $V_t$ is a unitary operator on $h_F$ $(t\in\mathbb{R}^+)$.

*Proposition 6.* The family of maps $\{\mathscr{S}_t, t\in\mathbb{R}^+\}$ from $B(h_0)$ to $B(h_0)$ given by

$$\mathscr{S}^t(X)=\iota^{-1}\circ\mathbb{E}_0^F(V_t\iota(X)V_t^\dagger),\qquad t\in\mathbb{R}^+\tag{5.4}$$

for each $X\in B(h_0)$ is a quantum dynamical semigroup on $B(h_0)$ with generator given by

$$\mathscr{M}(X)=\mathrm{i}[H_0, X]+L_0\theta_0X\theta_0L_0^\dagger-\tfrac{1}{2}\{L_0L_0^\dagger, X\}.\tag{5.5}$$

Let $\{\beta_t, t\in\mathbb{R}\}$ denote the unitary group of automorphisms of $B(h_F)$ given by

$$\beta_t = \mathrm{Ad}(I\hat{\otimes}\Gamma_F(S_t)), \qquad t\in\mathbb{R}$$

where $\Gamma_F$ is the fermion second quantisation functor.

The cocycle property

$$V_{ts} = V_s\beta_s(V_t) \tag{5.6}$$

follows by a slight generalisation of the proof of theorem 8.1 in [14].

Hence we obtain a unitary group of automorphisms $\hat{\mathcal{J}}^t$ of $B(h_F)$ by the prescription of (1.6) with $V_t$, $\beta_t$ replacing $U_t$, $\alpha_t$ (respectively).

We say that $(B(h_F), \hat{\mathcal{J}}^t, \iota^{-1}\circ\mathbb{E}_0^F)$ is a fermionic stochastic dilation of $(B(h_0), \mathcal{J}^t)$.

As in the boson case, we will retain this nomenclature in the case where $h_F$ is of the form $h_0\otimes\mathcal{H}$ with $\mathcal{H}$ isomorphic to antisymmetric Fock space over a direct sum of copies of $L^2(\mathbb{R})$.

Let $\{(B_t^\xi, B_t^{\xi\dagger}), t\in\mathbb{R}^+\}$ denote fermion Brownian motion of variance $\sigma^2=\cos 2\xi$ ($\xi\in[0,\pi/2]$) in the state $\tilde{\omega}'=\omega_0\otimes\omega_\xi$ where $\omega_\xi$ is an extremal universally invariant quasi-free state on $B(\Gamma_F(L^2(\mathbb{R})))$.

The process may be realised as operators in $\tilde{h}_F = h_0\otimes\Gamma_F(L^2(\mathbb{R}))\otimes\Gamma_F(\overline{L^2(\mathbb{R})})$ via the prescription

$$B_t^\xi = \cos\xi(I\hat{\otimes} b(\chi_{[0,t)})\hat{\otimes} I) + \sin\xi(I\hat{\otimes} I\hat{\otimes}\bar{b}^\dagger(\overline{\chi_{[0,t)}}))$$

with $\omega_\xi$ acting as $\langle\Omega_F\otimes\bar{\Omega}_F, \cdot\, \Omega_F\otimes\bar{\Omega}_F\rangle$ where $\bar{\Omega}_F$ is the vacuum vector in $\Gamma_F(\overline{L^2(\mathbb{R})})$.

The analogues of (5.4), (5.5) and (5.6) in this context were obtained in [35], the equations for the cocycle and semigroup generator being

$$\begin{aligned} \mathrm{d}V_t = V_t[&\iota(L_0)\,\mathrm{d}B_t^\xi - \iota(\rho_0(L_0^\dagger))\,\mathrm{d}B_t^{\xi\dagger} \\ &+\mathrm{i}\iota(H_0) - \tfrac{1}{2}\cos^2\xi\iota(L_0L_0^\dagger) - \tfrac{1}{2}\sin^2\xi\iota(\rho_0(L_0^\dagger L_0))\,\mathrm{d}t] \end{aligned} \tag{5.7}$$

$$\begin{aligned} \mathcal{M}(X) = \mathrm{i}[H_0, X] &+ \cos^2\xi(L_0\theta_0X\theta_0L_0 - \tfrac{1}{2}\{L_0^\dagger L_0^\dagger, X\}) \\ &+\sin^2\xi(\theta_0L_0^\dagger XL_0\theta_0 - \tfrac{1}{2}\{\theta_0L_0^\dagger L_0\theta_0, X\}). \end{aligned} \tag{5.8}$$

In (5.8), we may write for $\beta>0$

$$\cos^2\xi = 1/(1+\mathrm{e}^{-\beta}), \qquad \sin^2\xi = \mathrm{e}^{-\beta}/(1+\mathrm{e}^{-\beta})$$

which indicates the possibility of constructing stationary fermionic stochastic dilations in an analogous way to § 2.

We will say that a fermionic stochastic dilation is of zero (finite) temperature whenever the cocycle is a solution of (5.3) ((5.7)).

In the zero temperature case, we define a family of injections $\{\iota_t, t\in\mathbb{R}\}$ from $B(h_0)$ to $B(h_F)$ by

$$\iota_t = \mathcal{J}^t\circ\iota.$$

Then $\{B(h_F), \{\iota_t, t\in\mathbb{R}\}, \omega'\}$ is a quantum stochastic process and for each $X\in B(h_0)$

$$\mathrm{d}X_t = \iota_t\{(L_0\rho_0(X) - XL_0)\,\mathrm{d}B_t + (X\rho_0(L_0^\dagger) - \rho_0(L_0^\dagger X))\,\mathrm{d}B_t^\dagger + \mathcal{M}(X)\,\mathrm{d}t\}. \tag{5.9}$$

We cannot simplify (5.9) in an analogous way to (2.8) because of the non-definite parity of $U_t$ (e.g. $\iota_t(\mathrm{d}B_t)\neq\mathrm{d}B_t$ in general).

## 6. Fermion stochastic Bloch dilations, relaxation times and standard forms

Let $h_0 = \mathbb{C}$. We define a $Z_2$-grading on $h_0$ by taking $h_{0,+}$ and $h_{0,-}$ to be the linear spans of the vectors $\binom{1}{0}$ and $\binom{0}{1}$ respectively.

Hence we may consider fermionic stochastic Bloch dilations of $(M_2(\mathbb{C}), \mathcal{J}')$ where the semigroup $\mathcal{J}'$ satisfies the equation (3.1) and has a generator of the form (5.5) or (5.8). We begin by taking $\omega = 0$ and obtaining the fermion analogue of proposition 1, in the zero temperature case.

*Proposition 7.* A necessary and sufficient condition for (3.3) to hold is

$$L_0 \in \mathscr{R}_2. \tag{6.1}$$

Furthermore in this case, we obtain

$$\begin{aligned} \lambda_1 &= \tfrac{1}{2}(|\alpha+\delta|^2 + |\beta+\gamma|^2) \\ \lambda_2 &= \tfrac{1}{2}(|\alpha+\delta|^2 + |\beta-\gamma|^2) \\ \lambda_3 &= |\beta|^2 + |\gamma|^2 \end{aligned} \tag{6.2}$$

$$\begin{aligned} \varepsilon_1 &= (2/\lambda_1)(\mathrm{Re}\ \alpha\bar{\gamma} + \mathrm{Re}\ \bar{\alpha}\beta) \\ \varepsilon_2 &= (2/\lambda_2)(\mathrm{Im}\ \alpha\bar{\gamma} - \mathrm{Im}\ \alpha\bar{\beta}) \\ \varepsilon_3 &= (2/\lambda_3)(|\gamma|^2 - |\beta|^2). \end{aligned} \tag{6.3}$$

We observe that the relation (3.7) again holds for the inverse relaxation times.

Let $\lambda_j^\xi$, $\varepsilon_j^\xi$ denote the inverse relaxation times and equilibrium values, respectively, in the finite temperature case ($j = 1, 2, 3$). Repeating the computation of proposition 7, we obtain the following fermion analogues of equations (3.11) and (3.12), for $j = 1$, 2, 3

$$\lambda_j^\xi = \lambda_j \tag{6.4}$$

$$\varepsilon_j^\xi = \sigma^2 \varepsilon_j. \tag{6.5}$$

In this case, the variance is given by

$$\begin{aligned} \sigma^2 &= \cos^2\xi - \sin^2\xi \\ &= (1 - \mathrm{e}^{-\beta})/(1 + \mathrm{e}^{-\beta}) \\ &= \tanh \tfrac{1}{2}\beta \propto \beta. \end{aligned}$$

So, we see that, just as in the boson case, the equilibrium values are inversely proportional to the temperature of the dilation. In contrast to the boson case, however, the relaxation times are invariant with respect to temperature changes.

We now investigate the standard form of dilation, in the zero temperature case, with $H_0 = \frac{1}{2}\omega\sigma_z$ in (5.5). To carry out our analysis, we need the fermion analogue of equation (4.5).

Putting $L_0 = a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in (5.9), we obtain

$$\mathrm{d}a_t = \iota_t\{-[(\alpha+\delta)a + \gamma I]\,\mathrm{d}B_t + [(\bar{\alpha}+\bar{\delta})a - \bar{\beta}I]\,\mathrm{d}B_t^\dagger + \mathcal{M}(a)\,\mathrm{d}t\}. \tag{6.6}$$

Equation (6.6) generalises the fermion diffusion processes considered in [36], where $a_t$ was restricted to be an odd operator for all $t \in \mathbb{R}^+$.

Putting $\eta = \arg(\alpha+\delta)$, we make the gauge transformations $B_t \to e^{i\eta}B_t$, $B_t^\dagger \to e^{-i\eta}B_t^\dagger$ [14] and write $\gamma' = e^{-i\eta}\gamma$, $\beta' = e^{-i\eta}\beta$.

Let $P_t = -i(B_t - B_t^\dagger)$ $(t \in \mathbb{R}^+)$ then (6.6) may be written

$$da_t = \iota_t\{-i|\alpha+\delta|a\, dP_t - \gamma'\, dB_t - \beta'\, dB_t^\dagger + \mathcal{M}(a)\, dt\} \tag{6.7}$$

(cf equation (4.5)).

Let $\mathscr{A}$ denote the von Neumann algebra in $B(\Gamma_F(L^2(\mathbb{R})))$ generated by $\Psi(f) = a(f) - a^\dagger(\bar{f})$ $(f \in L^2(\mathbb{R}))$ and $m$ be the tracial state on $\mathscr{A}$ obtained by restriction of vacuum expectation. There is a unique Hilbert space isomorphism $E : \Gamma_F(L^2(\mathbb{R})) \to L^2(\mathscr{A})$ (the *duality transform* of [37]) such that for each $t \in \mathbb{R}^+$,

$$EP_tE^{-1} = \Phi_t \tag{6.8}$$

where

$$\Phi_t = -i\Psi(\chi_{[0,t)}).$$

$\{\Phi_t, t \in \mathbb{R}^+\}$ is the Clifford process [8, 38] in $\mathscr{A}$. It plays the role of a fermionic analogue of the classical Brownian motion process.

Inspection of (6.2) in the light of (6.1) indicates that there are three possibilities for obtaining a fermionic Bloch dilation in standard form. These are

(i)′ $\quad \alpha = \beta = \delta = 0, \qquad \gamma \neq 0$

(ii)′ $\quad \alpha = \gamma = \delta = 0, \qquad \beta \neq 0$

(iii)′ $\quad \beta = \gamma = 0, \qquad \alpha, \delta \neq 0.$

We examine each of these in turn

(i)′
$$da_t = \iota_t[-\gamma'\, dB_t + (i\omega a - \tfrac{1}{2}|\gamma|^2 a)\, dt]. \tag{6.9}$$

The cocycle satisfies the equation

$$dV_t^\dagger = V_t^\dagger \iota(\gamma' a^\dagger\, dB_t + \gamma' a\, dB_t^\dagger + \tfrac{1}{2}(i\omega[a, a^\dagger] - |\gamma'|^2 a^\dagger a)\, dt) \tag{6.10}$$

whence we see that each $V_t^\dagger$ is an even operator so (6.9) may be written

$$da_t = -\gamma'\, dB_t + (i\omega a_t - \tfrac{1}{2}|\gamma'|^2 a_t)\, dt \tag{6.11}$$

where $a_t = \iota_t(a)$ $(t \in \mathbb{R}^+)$.

Making the gauge transformation $B_t \mapsto \exp[i(\arg\gamma' - \pi/2)]B_t$ we may write the solution to (6.11) as the fermionic Ornstein–Uhlenbeck process ([35], see also [39])

$$a_t = \exp(i\omega - \tfrac{1}{2}|\gamma|^2 t)\iota(a_0) + |\gamma| \int_0^t \exp(i\omega - \tfrac{1}{2}|\gamma|^2)(t-\tau)\, dB_\tau. \tag{6.12}$$

The semigroup is given by

$$\mathcal{J}^t(A) = \begin{pmatrix} a_{11} & \exp[(i\omega - \frac{1}{2}|\gamma|^2)t]a_{12} \\ \exp[-(i\omega + \frac{1}{2}|\gamma|^2)t]a_{21} & a_{22} + [1 - \exp(-|\gamma|^2 t)](a_{11} - a_{22}) \end{pmatrix} \tag{6.13}$$

for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathbb{C})$, so this case describes a quasi-free relaxation.

We remark that the solution to (6.10) is explicitly constructed as a 'continuous product integral' in [14]

(ii)′
$$da_t = \iota_t(-\beta'\, dB_t^\dagger + (i\omega a - \tfrac{1}{2}|\beta'|^2 a)\, dt. \tag{6.14}$$

The cocycle satisfies the equation

$$\mathrm{d}V_t^{\downarrow} = V_t^{\downarrow}\iota(\beta' a\,\mathrm{d}B_t + \beta' a^{+}\,\mathrm{d}B_t^{\dagger} + \tfrac{1}{2}(\mathrm{i}\omega[a, a^{\dagger}] - |\beta'|^2 aa^{\dagger})\,\mathrm{d}t). \tag{6.15}$$

Again, each $V_t^{\downarrow}$ is an even operator so (6.14) may be written

$$\mathrm{d}a_t = [-\beta'\,\mathrm{d}B_t^{\dagger} + (\mathrm{i}\omega a_t - \tfrac{1}{2}|\beta'|^2 a_t)]\,\mathrm{d}t. \tag{6.16}$$

Making the gauge transformation $B_t \to \exp[\mathrm{i}(\arg\beta' - \pi/2)]B_t$, we obtain as the solution to (6.16) the fermion Ornstein-Uhlenbeck process

$$a_t = \exp[(\mathrm{i}\omega - \tfrac{1}{2}|\beta|^2)t]\iota(a_0) + |\beta| \int_0^t \exp[(\mathrm{i}\omega - \tfrac{1}{2}|\beta|^2)(t-\tau)]\,\mathrm{d}B_\tau^{+}. \tag{6.17}$$

The semigroup in this case is again a quasi-free relaxation, given by

$$\mathscr{T}^t(A) = \begin{pmatrix} a_{11} + [1 - \exp(-|\beta|^2 t)](a_{22} - a_{11}) & \exp[(\mathrm{i}\omega - \frac{1}{2}|\beta|^2)t]a_{12} \\ \exp[-(\mathrm{i}\omega + \frac{1}{2}|\beta|^2 t)]a_{21} & a_{22} \end{pmatrix}. \tag{6.18}$$

(iii)′ Applying (6.8) we consider this as an equation in $h_0 \otimes L^2(\mathscr{A})$

$$\mathrm{d}a_t = \iota_t[-\mathrm{i}|\alpha + \delta|a\,\mathrm{d}\Phi_t + (\mathrm{i}\omega a - \tfrac{1}{2}|\alpha + \delta|^2 a)\,\mathrm{d}t]. \tag{6.19}$$

A cocycle for the dilation is given by the solution of

$$\mathrm{d}V_t^a = V_t^a\iota[\tfrac{1}{2}|\alpha + \delta|\,\mathrm{d}\Phi_t + (\mathrm{i}\omega[a, a^{+}] - \tfrac{1}{4}|\alpha + \delta|^2)\,\mathrm{d}t] \tag{6.20}$$

which is

$$V_t^a = \exp[\tfrac{1}{2}\mathrm{i}|\alpha + \delta|\Phi_t + \mathrm{i}\omega\iota([a, a^{+}])t]. \tag{6.21}$$

Clearly, for $t > 0$, $V_t^a$ is not of definite parity.

The semigroup is given by

$$\mathscr{T}_a^t(A) = \begin{pmatrix} a_{11} & \exp[-(\frac{1}{2}|\alpha + \delta|^2 - \mathrm{i}\omega)t]a_{12} \\ \exp[-(\frac{1}{2}|\alpha + \delta|^2 + \mathrm{i}\omega)t]a_{21} & a_{22} \end{pmatrix} \tag{6.22}$$

and is a Larmor relaxation.

From (6.13), (6.18) and (6.22) we obtain the commutation relations:

$$\begin{aligned} &[\mathscr{T}_{\downarrow}^t, \mathscr{T}_a^t] = [\mathscr{T}_{\uparrow}^t, \mathscr{T}_a^t] = 0 \\ &[\mathscr{T}_{\downarrow}^t, \mathscr{T}_{\uparrow}^t] \neq 0 \end{aligned} \tag{6.23}$$

for all $t \in \mathbb{R}^+$.

We define $\mathscr{T}_{a\downarrow}^t = \mathscr{T}_a^t \circ \mathscr{T}_{\downarrow}^t$, $\mathscr{T}_{a\uparrow}^t = \mathscr{T}_a^t \circ \mathscr{T}_{\uparrow}^t$.

Let $\mathscr{N}_{\mathrm{F}} = \mathbb{C}^2 \otimes L^2(\mathscr{A}) \otimes \Gamma_{\mathrm{F}}(L^2(\mathbb{R}))$ and $k$ be the canonical injection of $M_2(\mathbb{C})$ into $B(h_{\mathrm{F}})$. We denote by $\mathbb{E}_0^{\mathrm{a}}$, conditional expectation with respect to the tracial state $m$ on $B(\mathbb{C}^2 \otimes L^2(\mathscr{A}))$.

Defining the automorphism groups $\tilde{\mathscr{T}}_{\uparrow}^t$, $\tilde{\mathscr{T}}_{\downarrow}^t$, $\tilde{\mathscr{T}}_a^t$ by analogy with their boson analogues in § 4, we obtain the following fermion version of theorem 2.

*Theorem 8.* $(B(\mathscr{N}_{\mathrm{F}}), \tilde{\mathscr{T}}_a^t \circ \tilde{\mathscr{T}}_{\downarrow}^t, k^{-1} \circ \mathbb{E}_0^{\mathrm{a}} \circ \mathbb{E}_0^{\mathrm{F}})$ and $(B(\mathscr{N}_{\mathrm{F}}), \tilde{\mathscr{T}}_a^t \circ \tilde{\mathscr{T}}_{\uparrow}^t, k^{-1} \circ \mathbb{E}_0^{\mathrm{a}} \circ \mathbb{E}_0^{\mathrm{F}})$ are fermion stochastic dilations of $(M_2(\mathbb{C}), \mathscr{T}_{a\downarrow}^t)$ and $(M_2(\mathbb{C}), \mathscr{T}_{a\uparrow}^t)$, respectively.

These dilations are described by the SDE obtained from putting $\beta = 0$ and $\gamma = 0$ (respectively) in (6.6). Since the group G is wholly contained in the even part of $M_2(\mathbb{C})$, the validity of theorem 3 in the fermion case follows immediately.

## 7. The general quantum Bloch dilation in standard form

From (4.10), (4.13); (4.17), (6.13), (6.18) and (6.22) we obtain the following commutation relations, in addition to (4.18) and (6.23)

$$\begin{aligned} &[\mathrm{P}_c^t, \mathscr{I}_a^t] = [\mathrm{P}_c^t, \mathscr{I}_\uparrow^t] = [\mathrm{P}_c^t, \mathscr{I}_\downarrow^t] = 0 \\ &[\mathscr{I}_a^t, \mathrm{P}_\uparrow^t] = [\mathscr{I}_a^t, \mathrm{P}_\downarrow^t] = 0 \\ &[\mathrm{P}_\uparrow^t, \mathscr{I}_\uparrow^t] = [\mathrm{P}_\downarrow^t, \mathscr{I}_\downarrow^t] = 0 \\ &[\mathrm{P}_\uparrow^t, \mathscr{I}_\downarrow^t] \neq 0, [\mathrm{P}_\downarrow^t, \mathscr{I}_\uparrow^t] \neq 0 \qquad \forall t \in \mathbb{R}^+. \end{aligned} \tag{7.1}$$

Equation (7.1) indicates the possibility of dilating the semigroups

$$Q_\uparrow^t = \mathrm{P}_c^t \circ \mathscr{I}_a^t \circ \mathrm{P}_\uparrow^t \circ \mathscr{I}_\uparrow^t$$

and (7.2)

$$Q_\downarrow^t = \mathrm{P}_c^t \circ \mathscr{I}_a^t \circ \mathrm{P}_\downarrow^t \circ \mathscr{I}_\downarrow^t$$

using a combination of boson and fermion noises.

We make the following definition, which generalises the boson and fermion structures defined in §§ 2 and 5.

Let $h_0$ be a $Z_2$-graded Hilbert space and $\mathscr{H}_Q$ be isomorphic to the tensor product of symmetric Fock space over a direct sum of copies of $L^2(\mathbb{R})$ with antisymmetric Fock space over a direct sum of copies of $L^2(\mathbb{R})$. (The direct sums need not be of the same cardinality; however either or both of them may be infinite.)

We write $\mathscr{N}_Q = h_0 \otimes \mathscr{H}_Q$. Let $l: B(h_0) \to B(h_Q)$ be the canonical injection given by $l(A) = A \otimes I_\mathrm{B} \hat{\otimes} I_\mathrm{F}$ where $I_\mathrm{B}$ and $I_\mathrm{F}$ are the identity operators on the symmetric and antisymmetric Fock spaces (respectively). Let $E_Q$ be a conditional expectation from $B(h_Q)$ to $l(B(h_0))$, $\{Q_t, t \in \mathbb{R}^+\}$ be a quantum dynamical semigroup in $B(h_0)$ and $\{\hat{Q}_t, t \in \mathbb{R}\}$ be a group of automorphisms of $B(\mathscr{N}_Q)$. We say that $(B(\mathscr{N}_Q), \hat{Q}_t, l^{-1} \circ \mathbb{E}_Q)$ is a quantum stochastic dilation of $(B(h_0), Q_t)$ whenever the following diagram commutes for all $t \in \mathbb{R}^+$.

$$\begin{array}{ccc} B(h_0) & \xrightarrow{Q_t} & B(h_0) \\ {\scriptstyle l}\downarrow & & \uparrow{\scriptstyle l^{-1}\circ \mathbb{E}_Q} \\ B(\mathscr{N}_Q) & \xrightarrow[\hat{Q}_t]{} & B(\mathscr{N}_Q) \end{array}$$

We take $h_0 = \mathbb{C}$ and

$$\mathscr{N}_Q = \mathbb{C}^2 \otimes L^2(\mathscr{C}, \mu) \otimes L^2(\mathscr{A}) \otimes \Gamma_\mathrm{B}(L^2(\mathbb{R})) \otimes \Gamma_\mathrm{A}(L^2(\mathbb{R})).$$

Let $\mathbb{E}_Q = \mathbb{E}_0^c \circ \mathbb{E}_0^a \circ \mathbb{E}_0^\mathrm{B} \circ \mathbb{E}_0^\mathrm{F}$ and define, for $t \in \mathbb{R}$,

$$\hat{Q}_\uparrow^t = \hat{\mathrm{P}}_c^t \circ \hat{\mathscr{I}}_A^t \circ \hat{\mathrm{P}}_\uparrow^t \circ \hat{\mathscr{I}}_\uparrow^t \qquad \hat{Q}_\downarrow^t = \hat{\mathrm{P}}_c^t \circ \hat{\mathscr{I}}_A^t \circ \hat{\mathrm{P}}_\downarrow^t \circ \hat{\mathscr{I}}_\downarrow^t. \tag{7.3}$$

The following result generalises theorems 2 and 8 and provides the most general quantum stochastic dilation of the Bloch equations in standard form via boson and fermion Brownian motions.

*Theorem 9.* $(B(\mathscr{N}_Q), \hat{Q}_\downarrow^t, l^{-1} \circ \mathbb{E}_Q)$, $(B(\mathscr{N}_Q), \hat{Q}_\uparrow^t, l^{-1} \circ \mathbb{E}_Q)$ are quantum stochastic dilations of $(M_2(\mathbb{C}), Q_\downarrow^t)$, $(M_2(\mathbb{C}), Q_\uparrow^t)$ (respectively).

Theorem 9 indicates the need for a boson-fermion stochastic calculus to be developed in the space $h_0 \otimes \Gamma_B(L^2(\mathbb{R})) \otimes \Gamma_A(L^2(\mathbb{R}))$.

Formally, we see that the relevant generalisation of the Itô product formula is given by the following table

| | $dA$ | $dA^\dagger$ | $dB$ | $dB^\dagger$ | $dt$ |
|---|---|---|---|---|---|
| $dA^\dagger$ | 0 | 0 | 0 | 0 | 0 |
| $dA$ | 0 | $dt$ | 0 | 0 | 0 |
| $dB^\dagger$ | 0 | 0 | 0 | 0 | 0 |
| $dB$ | 0 | 0 | 0 | $dt$ | 0 |
| $dt$ | 0 | 0 | 0 | 0 | 0 |

and for $H_0, K_0, L_0 \in B(h_0) H_0 = H_0^\dagger$, unitary cocycles are obtained from the solution of

$$dU_t = U_t\{l(K_0)\, dA_t - l(K_0^\dagger)\, dA_t^\dagger + l(L_0)\, dB_t - \rho(l(L_0))\, dB_t^\dagger$$
$$+[\mathrm{i}l(H_0) - \tfrac{1}{2} l(K_0 K_0^\dagger + L_0 L_0^\dagger)]\, dt\}. \tag{7.4}$$

To describe the situation in theorem 9, we require a further generalisation, i.e. the analogue of (7.4) in

$$\mathcal{N}_Q = \mathbb{C}^2 \otimes L^2(\mathscr{C}, \mu) \otimes L^2(\mathscr{A}) \otimes \Gamma_B(L^2(\mathbb{R})) \otimes \Gamma_F(L^2(\mathbb{R}))$$
$$\simeq \mathbb{C}^2 \otimes \Gamma_B(L^2(\mathbb{R})) \otimes \Gamma_B(L^2(\mathbb{R})) \otimes \Gamma_F(L^2(\mathbb{R})) \otimes \Gamma_F(L^2(\mathbb{R}))$$
$$\simeq \mathbb{C}^2 \otimes \Gamma_B(L^2(\mathbb{R}) \oplus L^2(\mathbb{R})) \otimes \Gamma_F(L^2(\mathbb{R}) \oplus L^2(\mathbb{R})).$$

Taking $H_0 = \frac{1}{2}\omega\sigma_z$ $(\omega \in \mathbb{R})$,

$$K_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad L_0 = \begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix} \in M_2(\mathbb{C})$$

in (7.4) and splitting $K_0$ and $L_0$ into their relevant summands, as in §§ 4 and 6, we find that the cocycles $W_t^\downarrow = U_t^c V_t^A U_t^\downarrow V_t^\downarrow$ and $W_t^\uparrow = U_t^c V_t^A U_t^\uparrow V_t^\uparrow$ giving rise to the unitary groups $Q_\downarrow^t$ and $Q_\uparrow^t$, respectively are the solutions of stochastic differential equations in $\mathcal{N}_Q$; the corresponding boson-fermion diffusion equations are

$$da_t^\downarrow = l_t^\downarrow(\mathrm{i}|\alpha - \delta| a\, dX_t - \mathrm{i}|\kappa + \nu| a\, d\Phi_t + \beta[a, a^\dagger]\, dA_t^\dagger$$
$$-\lambda\, dB_t^\dagger + (\mathrm{i}\omega - \tfrac{1}{2}(|\alpha + \delta|^2 + |\kappa + \nu|^2 + |\beta|^2 + |\lambda|^2)) a\, dt) \tag{7.5}$$

where $l_t^\downarrow = \hat{Q}_\downarrow^t \circ l$ $(t \in \mathbb{R}^+)$ and

$$da_t^\uparrow = l_t^\uparrow\{\mathrm{i}|\alpha - \delta| a\, dX_t - \mathrm{i}|\kappa + \nu| a\, d\Phi_t - \gamma[a, a^\dagger]\, dA_t - \mu\, dB_t + [\mathrm{i}\omega - \tfrac{1}{2}(|\alpha - \delta|^2$$
$$+|\kappa + \nu|^2 + |\gamma|^2 + |\mu|^2)] a\, dt\} \tag{7.6}$$

where $l_t^\uparrow = \hat{Q}_\uparrow^t \circ l$ $(t \in \mathbb{R}^+)$.

The results of this section generalise easily to the finite temperature case where the possibility of constructing stationary stochastic dilations, using the techniques of § 2, is available.

The rigorous development of the boson-fermion stochastic calculus may be established using the general techniques of [40]. Details will be published elsewhere.

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## Appendix. Bloch dilations via Poisson processes

We conclude this paper by examining the effect of introducing the gauge process of [1] into our dilation scheme.

For $T \in B(L^2(\mathbb{R}))$, we define the operator $\lambda(T)$ on $\Gamma_B(L^2(\mathbb{R}))$ through its action on the set of exponential vectors $\{\psi(f), f \in L^2(\mathbb{R})\}$

$$\lambda(T)\psi(f) = \frac{d}{d\varepsilon}\psi(e^{\varepsilon T}f)|_{\varepsilon=0}. \tag{A1}$$

The gauge process in $h_B$, $\{\Lambda_t, t \in \mathbb{R}^+\}$ is defined by

$$\Lambda_t = I \otimes \lambda(M_{\chi[0,t]}) \tag{A2}$$

where $M_{\chi[0,t]}$ is the operator of multiplication by $\chi_{[0,t]}$.

Let $W_0$ be a unitary operator on $\mathbb{C}^2$ so that

$$W_0 = \begin{pmatrix} \exp[i(\phi+k)]\cos\eta & \exp[i(\phi+\rho)]\sin\eta \\ -\exp[i(\phi-\rho)]\sin\eta & \exp[i(\phi-k)]\cos\eta \end{pmatrix} \tag{A3}$$

for $\phi$, $k$, $\rho$, $\eta \in [0, 2\pi]$. Let $W = W_0 \otimes I$.

By theorem 7.1 of [1], there exists a unique solution of the SDE

$$\begin{aligned} dU_t &= U_t((W-I)\,d\Lambda_t + L\,dA_t - WL^+\,dA_t^+ + (iH - \tfrac{1}{2}LL^+)\,dt) \\ U_0 &= I \end{aligned} \tag{A4}$$

with each $U_t$ a unitary operator in $h_B$. By imitating the argument of theorem 7.1 of [2], it is easy to see that these operators satisfy the cocycle condition (2.5). The prescription (2.3) again yields a quantum dynamical semigroup whose generator takes the same form (2.4) as that obtained by putting $W = I$ in (A4) [1]. In general, therefore, the gauge process appears only to introduce an element of redundancy into our scheme. However a dilation of some interest is obtained by putting $L_0 = l^{1/2}(W_0 - I)$ and $H_0 = -\frac{1}{2}i(W_0 - W_0^+) + \frac{1}{2}\omega\sigma_z$ in (A3) for $l \in \mathbb{R}^+$.

We thus obtain a cocycle satisfying

$$dU_t = U_t[(W-I)\,d\Pi_t + \tfrac{1}{2}i\omega j(\sigma_z)\,dt] \tag{A5}$$

where $\{\Pi_t^l, t \in \mathbb{R}^+\}$ is the Poisson process of intensity $l$ which satisfies the SDE [1]

$$\begin{aligned} d\Pi_t^l &= d\Lambda_t + \sqrt{l}(dA_t + dA_t^+) + l\,dt \\ \Pi_0^l &= 0. \end{aligned} \tag{A6}$$

From (2.4), we find that $U_t$ yields a dilation of the semigroup whose generator is given by

$$\mathscr{L}(X) = l(W_0XW_0^+ - X) + \mathrm{i}[\tfrac{1}{2}\omega\sigma_z, X] \tag{A7}$$

for $X \in M_2(\mathbb{C})$, which, we remark, may also be dilated via the cocycle satisfying

$$\mathrm{d}V_t = V_t[\sqrt{l}\,W\,\mathrm{d}A_t - \sqrt{l}\,W^+\,\mathrm{d}A_t^+ - \tfrac{1}{2}(lI - \tfrac{1}{2}\mathrm{i}\omega j(\sigma_z))\,\mathrm{d}t].$$

From (3.4) and (A7) we see that (A5) yields a bosonic stochastic Bloch dilation if and only if $W_0 \in \mathscr{R}_2$. From (A3) we obtain

$$\eta = \tfrac{1}{2}m\pi, \qquad k = \tfrac{1}{2}n\pi, \qquad \rho = \tfrac{1}{2}p\pi$$

where $m, n, p \in \mathbb{Z}$. Thus, we have four possible forms for $W_0$.

(a) $m$ even, $n$ even

$$W_0 = \mathrm{e}^{\mathrm{i}\phi}\begin{pmatrix} (-1)^{(m+n)/2} & 0 \\ 0 & (-1)^{(m-n)/2} \end{pmatrix}$$
$$= \pm \mathrm{e}^{\mathrm{i}\phi} I.$$

In this degenerate case, the dissipative part of $\mathscr{L}$ in (A7) vanishes.

(b) $m$ even, $n$ odd

$$W_0 = \mathrm{e}^{\mathrm{i}\phi}(-1)^{(m+n-1)/2}\mathrm{i}\sigma_z.$$

This is in standard form with parameters

$$\lambda_1 = \lambda_2 = 2l, \qquad \lambda_3 = 0 \qquad \text{and} \qquad \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0.$$

(c) $m$ odd, $p$ even

$$W_0 = \mathrm{e}^{\mathrm{i}\phi}(-1)^{(p+m-1)/2}\sigma_y.$$

This is characterised by the parameters $\lambda_1 = \lambda_3 = 2l$, $\lambda_2 = 0$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$ and is in standard form only in the degenerate case $l = 0$, when the dissipative part vanishes.

(d) $m$ odd, $p$ odd

$$W_0 = \mathrm{e}^{\mathrm{i}\phi}(-1)^{(p+m)/2}\sigma_x.$$

In this case we have $\lambda_2 = \lambda_3 = 2l$, $\lambda_1 = 0$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$. Again, we have a standard form only in the degenerate case $l = 0$. For our purpose, the most interesting case is (b). The semigroup is given by

$$\mathrm{P}^l_{(\mathrm{b})}(A) = \begin{pmatrix} a_{11} & \mathrm{e}^{-2lt}a_{12} \\ \mathrm{e}^{-2lt}a_{21} & a_{22} \end{pmatrix}$$

(where we have taken $\omega = 0$) and clearly commutes with each of $Q_\downarrow^t$ and $Q_\uparrow^t$. Hence, we may extend the scheme of theorem 9 by introducing another copy of $\Gamma_{\mathrm{B}}(L^2(\mathbb{R}))$ to accomodate the Poisson process of intensity $l$. Indeed, since the semigroups $\mathrm{P}^l_{(\mathrm{b})}$ commute for different values of $l \in [0, \infty]$ we may introduce $\Gamma_{\mathrm{B}}(\bigoplus_{j=1}^N L^2(\mathbb{R}))$ to take care of a finite number of Poisson processes of different intensities $l_j (l \leqslant j \leqslant N)$ where $N \in \mathbb{N}$. The further extension, to a countably infinite number of processes, will be dealt with elsewhere.

The generators of the semigroups in (b), (c) and (d) are, respectively, for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathbb{C})$ and taking $\omega = 0$,

$$\mathscr{L}_{(b)}(A) = \begin{pmatrix} a_{11} & -2la_{12} \\ -2la_{21} & a_{22} \end{pmatrix}$$

$$\mathscr{L}_{(c)}(A) = l\begin{pmatrix} a_{22} - a_{11} & -a_{21} - a_{12} \\ -a_{12} - a_{21} & a_{11} - a_{22} \end{pmatrix}$$

$$\mathscr{L}_{(d)}(A) = l\begin{pmatrix} a_{22} - a_{11} & a_{21} - a_{12} \\ a_{12} - a_{21} & a_{11} - a_{22} \end{pmatrix}$$

and these satisfy the relation

$$\mathscr{L}_{(c)}(A) - \mathscr{L}_{(d)}(A) = \mathscr{L}_{(b)}(A^t)$$

for all $A \in M_2(\mathbb{C})$, where $A^t$ denotes the transpose of the matrix $A$, and the commutation relations

$$[\mathscr{L}_{(b)}, \mathscr{L}_{(c)}] = [\mathscr{L}_{(b)}, \mathscr{L}_{(d)}] = [\mathscr{L}_{(c)}, \mathscr{L}_{(d)}] = 0.$$

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